

# Sphere packing bounds in the Grassmann and Stiefel manifolds

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**Abstract**—Applying the Riemann geometric machinery of volume estimates in terms of curvature, bounds for the minimal distance of packings/codes in the Grassmann and Stiefel manifolds will be derived and analyzed. In the context of space time block codes this leads to a monotonically increasing minimal distance lower bound as a function of the block length. This advocates large block lengths for the code design.

**Index Terms**—Sphere packings, space-time codes, Gilbert-Varshamov/Hamming bounds, Stiefel/Grassmann manifold

## I. INTRODUCTION

This work is inspired by Barg and Nogin's paper [1] for asymptotic packing bounds in the Grassmann manifold, based on an asymptotic expression for the volume of metric balls. The basic estimates defining the bounds are given by the well known Gilbert–Varshamov and Hamming (or sphere packing) inequalities: In a compact manifold  $M$  without boundary furnished with a topological metric  $d$ , let us denote the volume of the metric ball of radius  $\delta$  as  $\text{vol } B_d(\delta)$  (this quantity is presupposed to be independent of its center). Then for any given  $d_0$  there exists a packing (or code)  $\mathcal{C} \subset M$  with the prescribed minimal distance  $d_0$  and cardinality  $|\mathcal{C}|$  such that

$$\frac{\text{vol } M}{|\mathcal{C}|} \leq \text{vol } B_d(d_0) \quad (\text{Gilbert-Varshamov}) \quad (1)$$

while for any packing/code  $\mathcal{C} \subset M$  with data  $(d_0, |\mathcal{C}|)$

$$\text{vol } B_d\left(\frac{1}{2}d_0\right) \leq \frac{\text{vol } M}{|\mathcal{C}|} \quad (\text{Hamming}) \quad (2)$$

holds.

Taking for  $M$  the complex Grassmann manifold  $G_{k,n}^{\mathbb{C}}$  of  $k$  dimensional complex subspaces of  $\mathbb{C}^n$ , Barg and Nogin derived closed form expressions

$$\text{vol } B_d(\delta) = \begin{cases} \left(\sin \frac{\delta}{\sqrt{k}}\right)^{2nk+o(n)} & (\text{geodesic distance}) \\ \left(\frac{\delta}{\sqrt{k}}\right)^{2nk+o(n)} & (\text{chordal distance}) \end{cases} \quad (3)$$

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as  $n \rightarrow \infty$ , leading to

$$\sqrt{k} \arcsin\left(\frac{1}{\sqrt{2^{R/k}}}\right) \lesssim d_0 \lesssim 2\sqrt{k} \arcsin\left(\frac{1}{\sqrt{2^{R/k}}}\right) \quad (4)$$

$$\sqrt{\frac{k}{2^{R/k}}} \lesssim d_0 \lesssim \sqrt{2k \left(1 - \left(1 - \frac{1}{2^{R/k}}\right)^2\right)} \quad (5)$$

for geodesic, respectively chordal distance (defined later on), whereas  $R$  denotes the rate

$$R = \frac{1}{n} \log_2 |\mathcal{C}| \quad (6)$$

Furthermore Han and Rosenthal [2] recently derived upper bounds on the minimal distance (more general: on the diversity of space time codes) for packings on the unitary group  $U(n)$ .

A general capacity and performance analysis of space time codes in Rayleigh flat fading MIMO scenarios without channel state information at the transmitter [3], [4], [5], [6] revealed that the appropriate coding spaces are indeed

- the (scaled) complex Grassmann manifold  $G_{k,n}^{\mathbb{C}}$  (set of  $k$  dimensional linear subspaces of  $\mathbb{C}^n$ ), if the channel is unknown at the receiver
- the (scaled) complex Stiefel manifold  $V_{k,n}^{\mathbb{C}}$  (set of  $k$  orthonormal vectors in  $\mathbb{C}^n$ ) if the channel is known at the receiver.

Here  $k$  corresponds to the number of transmit antennas and  $n$  to the block length of the codes and the work in [1] refers to  $G_{k,n}^{\mathbb{C}}$  as  $n \rightarrow \infty$  while [2] refers to  $V_{k,n}^{\mathbb{C}}$  as  $k = n$ .

The aim of this work is to close the gap between those two results by deriving bounds on the minimal distance for codes/packings in  $G_{k,n}^{\mathbb{C}}, V_{k,n}^{\mathbb{C}}$  for arbitrary  $(k, n)$  (section III): Applying the bounds (1), (2) with equality, the main task is to solve the equation

$$B_d(\delta) = c, \quad c \in \mathbb{R} \quad (7)$$

for (minimal) distances  $\delta$  in  $G_{k,n}^{\mathbb{C}}, V_{k,n}^{\mathbb{C}}$ , with respect to some appropriate distance measure  $d$ . To this end volume estimates for the volume of (small) balls  $B_d(\delta)$  induced by curvature bounds for  $G_{k,n}^{\mathbb{C}}$  and  $V_{k,n}^{\mathbb{C}}$  come into play. Associated comparison spaces with constant curvature and simple volume forms provide bounds for  $B_d(\delta)$ . In particular the lower bound turns out to permit a simple closed form expression with respect to  $(k, n)$ . Its analysis culminates in Theorem IV.1 for the geodesic minimal distance lower bound and Corollary IV.2 for the minimal distance  $\tilde{d}_0$  of the corresponding space time codes. Surprisingly it turns out, that the minimal distance  $\tilde{d}_0$

grows at least proportional to  $\sqrt{n}$ , while keeping the rate and the transmit power per time step constant. That is, increasing the block length enhances the possible minimal distance, thus in coding spaces with large block lengths there exists codes with potentially better error performance than in 'small' coding spaces. Since most of the space time coding research efforts in the literature deal with small dimensional coding spaces such as  $U(k)$  (e.g. [6]), future research in the more general  $G_{k,n}^{\mathbb{C}}$ ,  $V_{k,n}^{\mathbb{C}}$  promises performance gains.

Apart from space time codes recent developments in the design of space frequency codes [7], [8] also indicate that the relevant coding spaces are subspaces of large dimensional Stiefel and Grassmann manifolds. Thus the achieved results here may be of considerable importance for space frequency code design.

This article proceeds as follows. Section II deals with notational conventions and basic definitions concerning the Stiefel and Grassmann manifolds (the coding spaces for space time or space frequency codes). In section III explicit bounds for the minimal distance will be calculated and compared to results obtained elsewhere. Further analysis on the lower bound will be performed in section IV, culminating in Theorem IV.1. Its implications for the minimal distance in coding theory will be pointed out in Corollary IV.2. Finally section V gives a summary of the results.

## II. THE COMPLEX STIEFEL AND GRASSMANN MANIFOLDS

The complex Stiefel and Grassmann manifolds together with their topological metrics (coding distance function in the language of coding theory) considered in this work constitute the focus of this section. For the analysis in later sections we also need some explicit curvature computations and rigorous proofs, which can be found in the appendices A and B.

Readers who are mainly interested in the results concerning packings/coding and who are willing to accept the (quite standard) differential geometric facts can read this section without reference to the appendices, where further details can be found.

A survey of the geometry of the *real* Stiefel and Grassmann manifolds aimed at non-specialists can be found in [9]<sup>1)</sup> and for an elementary introduction to differential geometric concepts see e.g. [10].

### A. The Stiefel manifold $V_{k,n}^{\mathbb{C}}$

The (complex) Stiefel manifold

$$V_{k,n}^{\mathbb{C}} := \{\Phi \in \mathbb{C}^{n \times k} \mid \Phi^\dagger \Phi = \mathbf{1}\} \quad (8)$$

( $\mathbf{1}$  denotes the identity matrix) can be equipped with the structure of an  $U(n)$ -normal homogeneous space, which justifies the coset representation

$$V_{k,n}^{\mathbb{C}} \cong U(n) / (\mathbf{1}_{U(n-k)} \mathbf{0}) \quad , \quad \Phi \cong \bar{\Phi} \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} \quad (9)$$

( $\bar{\Phi} \in U(n)$ ), in particular

$$\dim_{\mathbb{C}} V_{k,n}^{\mathbb{C}} = \dim_{\mathbb{C}} U(n) - \dim_{\mathbb{C}} U(n-k) = k(n - \frac{k}{2}) \quad (10)$$

<sup>1)</sup>The complex case considered here is similar to the real case but in some places certain peculiarities of the complex structure come into play

and

$$\begin{aligned} \text{vol } V_{k,n}^{\mathbb{C}} &= \text{vol } U(n) / \text{vol } U(n-k) \\ &= \prod_{i=n-k+1}^n |S^{2i-1}| = \prod_{i=n-k+1}^n \frac{2\pi^i}{(i-1)!} \end{aligned} \quad (11)$$

For  $V_{k,n}^{\mathbb{C}}$  as a Riemannian manifold the concept of geodesics and geodesic distance can be applied to obtain a canonical distance measure  $r^V$ : Denoting the tangent space of the unitary group  $U(n)$  by  $\mathfrak{u}(n)$  consisting of skew-Hermitian  $n$ -by- $n$  matrices, tangents of  $V_{k,n}^{\mathbb{C}}$  may be represented as

$$\mathfrak{u}(n) \ni X = \begin{pmatrix} A & -B^\dagger \\ B & \mathbf{0} \end{pmatrix} \quad , \quad A \in \mathfrak{u}(k), B \in \mathbb{C}^{(n-k) \times k} \quad (12)$$

and

$$(r^V)^2 = \frac{1}{2} \|X\|_{\text{F}}^2 = \frac{1}{2} \|A\|_{\text{F}}^2 + \|B\|_{\text{F}}^2 \quad (13)$$

is the squared geodesic length of the geodesic connecting  $\Psi = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} \in V_{k,n}^{\mathbb{C}}$  with  $\Phi = (\exp X) \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} \in V_{k,n}^{\mathbb{C}}$ . Here  $\exp$  denotes the matrix exponential and the geodesic distance between arbitrary points  $\Psi', \Phi' \in V_{k,n}^{\mathbb{C}}$  follows from the isometric transformation  $\Psi = \Psi'^{-1} \Psi'$  and  $\Phi = \Phi'^{-1} \Phi'$ .

The canonical embedding (8) of  $V_{k,n}^{\mathbb{C}}$  into the vector space  $(\mathbb{C}^{n \times k}, \langle \cdot, \cdot \rangle_{\mathbb{C}})$  motivates the definition of another topological ('chordal') metric/distance

$$d^V(\Phi, \Psi) := \|\Phi - \Psi\|_{\text{F}} \quad , \quad \Phi, \Psi \in V_{k,n}^{\mathbb{C}} \quad (14)$$

which is important for space time coding, where it represents the decision criterion at the maximum-likelihood receiver, if the channel is known at the receiver ('coherent' case), see [4], [6]. Note, that  $d^V$  is entirely different from the geodesic distance  $r^V$ . Nevertheless we have<sup>2)</sup>

### Proposition II.1

For  $k = n$  or  $k \leq \frac{n}{2}$  the metrics  $d^V$  and  $r^V$  are locally equivalent, thus in sufficiently small neighborhoods there exist constants  $\alpha^V > 0$ ,  $\beta^V > 0$  such that

$$\beta^V d^V \leq r^V \leq \alpha^V d^V \quad (15)$$

holds

This equivalence links the abstract (geodesic) sphere packing problem to space time coding. The restriction to the cases  $k = n$  and  $k \leq \frac{n}{2}$  is mainly for convenience, since the main analysis will concentrate on  $k \ll n$ .

*Proof:* Lemma B.1, B.2, B.3 in Appendix B ■

### Remark II.2

While it is an easy exercise to find  $\beta^V = \frac{1}{\sqrt{2}}$  (Lemma B.1), no concrete values for  $\alpha^V$  have been obtained rigorously. However, for  $k = 2$ ,  $n = 4, 6, 8$  numerical simulations led to  $\alpha^V \approx \frac{\pi}{2 \cdot 0.9}$ .

<sup>2)</sup>At first sight the proposition seems obvious, but one has to take into account that  $d^V$  is expressed in terms of  $\Phi, \Psi \in V_{k,n}^{\mathbb{C}}$ , while  $r^V$  is expressed in terms of the space of tangents and these two spaces are linked by the matrix exponential which can not be written in closed form compare Appendix B. Furthermore unlike  $d^V$ ,  $r^V$  is NOT induced by (geodesics with respect to) the seemingly canonical embedding  $V_{k,n}^{\mathbb{C}} \subset \mathbb{C}^{n \times k}$ , compare Appendix A-I.2

### B. The Grassmann manifold $G_{k,n}^{\mathbb{C}}$

The (complex) Grassmann manifold

$$G_{k,n}^{\mathbb{C}} := \{\langle \Phi \rangle \mid \Phi \in V_{k,n}^{\mathbb{C}}\} \quad (16)$$

of all  $k$ -dimensional linear subspaces  $\langle \Phi \rangle$  of  $\mathbb{C}^n$  also carries the structure of a  $U(n)$ -normal homogeneous space with coset representation

$$G_{k,n}^{\mathbb{C}} \cong U(n) / \left( \begin{smallmatrix} U(k) & \mathbf{0} \\ \mathbf{0} & U(n-k) \end{smallmatrix} \right), \quad \langle \Phi \rangle \cong \Phi \Phi^1^{-1} \quad (17)$$

(with  $\Phi^1 := (\mathbf{1}, \mathbf{0})\Phi$ ) and

$$\dim_{\mathbb{C}} G_{k,n}^{\mathbb{C}} = k(n-k) \quad (18)$$

The total volume of  $G_{k,n}^{\mathbb{C}}$  is

$$\begin{aligned} \text{vol } G_{k,n}^{\mathbb{C}} &= \text{vol } V_{k,n}^{\mathbb{C}} / \text{vol } U(k) \\ &= \prod_{i=n-k+1}^n \frac{2\pi^i}{(i-1)!} \sqrt{\prod_{j=1}^k \frac{2\pi^j}{(j-1)!}} \end{aligned} \quad (19)$$

Tangents become

$$X = \begin{pmatrix} \mathbf{0} & -B^{\dagger} \\ B & \mathbf{0} \end{pmatrix}, \quad B \in \mathbb{C}^{(n-k) \times k} \quad (20)$$

with squared geodesic length  $\frac{1}{2}\|X\|_{\text{F}}^2 = \|B\|_{\text{F}}^2$ , but there is an alternative notation in terms of the vector of principal angles  $\vartheta$  between subspaces: To simplify matters let us assume  $k \leq n/2$  whenever we are in contact with the Grassmann manifold. This is no restriction, since for  $k \geq n/2$  we can always switch to the orthogonal complement. Then there are precisely  $k$  principal angles  $\vartheta_i$  between the subspaces  $\langle (\frac{1}{0}) \rangle$  and  $\langle (\exp X)(\frac{1}{0}) \rangle$ . Performing a singular value decomposition on the tangents (20) one obtains (compare A-I.3).  $\|B\|_{\text{F}} = \|\vartheta\|_{L^2}$  thus the geodesic distance  $r^G$  between  $\langle (\frac{1}{0}) \rangle$  and  $\langle (\exp X)(\frac{1}{0}) \rangle$  reads

$$r^G = \frac{1}{\sqrt{2}}\|X\|_{\text{F}} = \|B\|_{\text{F}} = \|\vartheta\|_{L^2} \quad (21)$$

As for  $V_{k,n}^{\mathbb{C}}$  there is also a different distance measure  $d^G$  in  $G_{k,n}^{\mathbb{C}}$  induced by the maximum-likelihood receiver, which can be derived from the following geometric picture:

*Spherical embedding:* Unlike for the Stiefel manifold, there is no canonical embedding of  $G_{k,n}^{\mathbb{C}}$  into Euclidean space unless choosing a representing unitary frame  $\Phi_0$  in each subspace  $\langle \Phi \rangle \in G_{k,n}^{\mathbb{C}}$ . Nevertheless there exists an interesting embedding of  $G_{k,n}^{\mathbb{C}}$  into Euclidean space given in [11]: For  $\Phi \in V_{k,n}^{\mathbb{C}}$  there is a well-defined associated orthogonal projection

$$P_{\Phi} := \Phi \Phi^{\dagger} : \mathbb{C}^n \longrightarrow \langle \Phi \rangle \quad (22)$$

of norm  $\|P - k/n \mathbf{1}\|_{\text{F}}^2 = k(n-k)/n$  and  $\text{tr } P = k$ , which justifies the embedding

$$\begin{aligned} G_{k,n}^{\mathbb{C}} &\hookrightarrow S^{n^2-2}(\sqrt{k(n-k)/n}) \subset \mathbb{R}^{n^2-1}, \\ \langle \Phi \rangle &\longmapsto P_{\Phi} - \frac{k}{n} \mathbf{1} \end{aligned} \quad (23)$$

This motivates the 'chordal' topological metric

$$d^G(\langle \Phi \rangle, \langle \Psi \rangle) := \|\sin \vartheta\|_{L^2} = \frac{1}{\sqrt{2}}\|P_{\Phi} - P_{\Psi}\|_{\text{F}} \quad (24)$$

( $\Phi, \Psi \in V_{k,n}^{\mathbb{C}}$ ). Comparing  $d^G$  with the geodesic distance  $r^G$  (21) between two subspaces we observe

### Proposition II.3

$$\beta^G d^G \leq r^G \leq \alpha^G d^G \quad (25)$$

whereas  $\beta^G = 1$  and  $\alpha^G = \frac{\pi}{2}$ .

### III. BOUNDS FOR THE MINIMAL DISTANCE

Now let us specialize the general packing/coding bounds (1), (2). Set

$$(M, d) := \begin{cases} (V_{k,n}^{\mathbb{C}}, d^V) \\ (G_{k,n}^{\mathbb{C}}, d^G) \end{cases} \quad (26)$$

and (compare (10), (18))

$$D := \dim_{\mathbb{R}} M = \begin{cases} k(2n-k), & M = V_{k,n}^{\mathbb{C}} \\ 2k(n-k), & M = G_{k,n}^{\mathbb{C}} \end{cases} \quad (27)$$

for the two cases of interest. In the sequel other symbols like  $\alpha$  are used generically to denote  $\alpha^V$  or  $\alpha^G$  when specialized to the corresponding spaces  $V_{k,n}^{\mathbb{C}}$ ,  $G_{k,n}^{\mathbb{C}}$ . Denote by

$$v(r) := \text{vol } B(r) \quad (28)$$

the volume of the geodesic ball of radius  $r$  in  $M$ , which is independent of its center by left invariance of the Riemannian metric. With this notation the Gilbert-Varshamov (1) and Hamming bound (2) for packings  $\mathcal{C}$  on  $M$  can be compactly rewritten as

$$\underline{r}_0 := v^{-1}\left(\frac{\text{vol } M}{2^{nR}}\right) \leq r_0 \leq 2v^{-1}\left(\frac{\text{vol } M}{2^{nR}}\right) =: \bar{r}_0 \quad (29a)$$

or relaxed w.r.t. coding distances  $d_0$  using (15), (25)

$$\underline{d}_0 := \frac{1}{\alpha}v^{-1}\left(\frac{\text{vol } M}{2^{nR}}\right) \leq d_0 \leq \frac{2}{\beta}v^{-1}\left(\frac{\text{vol } M}{2^{nR}}\right) =: \bar{d}_0 \quad (29b)$$

So packing bounds are related to the coding bounds by simply setting  $\alpha = \beta = 1$ , thus replacing the (topological) metric distances by geodesic distances. Due to the rather difficult to obtain explicit value for  $\alpha^V$  in (15) we focus on the packing bounds (29a) for most of the remaining analysis, keeping in mind the simple relationship between statements about packings and statements about space time coding.

To obtain the desired bounds for the minimal distance provided by (29) we need closed form expressions for the volume  $v$  of small balls in  $M$ . As has been already indicated in the introduction, this is a difficult task in general: The canonical volume forms on  $G_{k,n}^{\mathbb{C}}$  and  $V_{k,n}^{\mathbb{C}}$  are elaborate to calculate. Alternatively a common tool to compute volumes in Riemannian geometry arises from curvature, using Jacobi vector fields (see e.g. [12] for details). Unfortunately a direct application can not be performed since we would have needed a diagonalization of  $XY - YX$  for each horizontal (compare A-I.1)  $\|X\| = \|Y\| = 1$  in  $u(n)$  written in closed form. But there are simple volume estimates which will be presented in III-A. In III-B the results will be compared to those already obtained in [1], [2], in a few (computational simple) cases.

### A. Bishop/Günther volume bounds

The method for volume computations in Riemannian manifolds using Jacobi vector fields can be looked up in [12, theorem 3.101]. For  $\kappa \in \mathbb{R}$  let

$$v^\kappa(r) := \left( \frac{1}{\sqrt{\kappa}} \right)^{D-1} |S^{D-1}| \int_0^r (\sin \sqrt{\kappa} t)^{D-1} dt \quad (30)$$

denote the volume of the geodesic ball of radius  $r$  in the manifold of constant curvature  $\kappa$  and let  $\underline{\kappa} \leq \kappa \leq \bar{\kappa}$  be defined by (compare (A.5), (A.4))

$$\begin{aligned} \underline{\kappa} &:= \frac{1}{D-1} \min_{e_i} \text{Ric}(e_i, e_i) \\ \bar{\kappa} &:= \max_{\|X\| = \|Y\| = 1} K(X, Y) \end{aligned} \quad (31)$$

then we obtain monotone volume bounds  $v_l(r) \leq v(r) \leq v_u(r)$  for arbitrary  $0 \leq r \leq \frac{\pi}{\sqrt{\bar{\kappa}}}$  by

$$v_l(r) := v^{\bar{\kappa}}(r), \quad v_u(r) := v^{\underline{\kappa}}(r) \quad (32)$$

From  $\lambda \leq \kappa \implies v^\kappa(r) \leq v^\lambda(r)$  and  $K \geq 0$  in  $M$  (A.6) we can further relax  $\underline{\kappa}$  to zero, which yields the simple upper volume bound

$$v_u(r) = v^0(r) = |B^D| r^D \quad (33)$$

A lower volume bound comes from an upper bound  $\bar{\kappa}$  for  $K$ . Inserting tangents  $X, Y$  (12), (resp. (20)) into (A.4) subject to  $\|X\| = \|Y\| = 1$  yields

$$K(X, Y) \leq \bar{\kappa} = \begin{cases} 2 & (U(k) = V_{k,k}^{\mathbb{C}}) \\ \frac{5}{2} & (V_{k,n}^{\mathbb{C}}, k < n) \\ 4 & (G_{k,n}^{\mathbb{C}}) \end{cases} \quad (34)$$

Plugging this bounds into (29a) we end up with

$$r_0 = (v^0)^{-1} \left( \frac{\text{vol } M}{2^{nR}} \right) \leq r_0 \leq (v^{\bar{\kappa}})^{-1} \left( \frac{\text{vol } M}{2^{nR}} \right) = \bar{r}_0 \quad (35)$$

With these settings an explicit (Maple-) calculation revealed

$k \setminus n$	$k$	$2k$	$3k$	$4k$
1	[1.57, 3.14] {0.500, 1.05}	[1.06, 3.49] {0.595, 1.40}	[0.941, -1] {0.630, 1.71}	[0.886, -1]
2	[1.58, -1] {0.771, -1}	[1.38, -1] {0.909, -1}	[1.32, -1] {0.973, -1}	[1.29, -1]
3	[1.74, -1] {0.977, -1}	[1.66, -1] {1.15, -1}	[1.63, -1] {1.24, -1}	[1.61, -1]
4	[1.92, -1] {1.15, -1}	[1.92, -1] {1.36, -1}	[1.89, -1] {1.46, -1}	[1.88, -1]

$[r_0, \bar{r}_0]$  for  $V_{k,n}^{\mathbb{C}}$  and  $\{r_0, \bar{r}_0\}$  for  $G_{k,n}^{\mathbb{C}}$  with respect to (35) for  $R = 1$

$k \setminus n$	$k$	$2k$	$3k$	$4k$
1	[0.0031, 0.0061] {0.001, 0.002}	[0.0165, 0.0330] {0.0055, 0.0110}	[0.0223, 0.0446] {0.0098, 0.0197}	[0.0251, 0.0502]
2	[0.0700, 0.140] {0.0341, 0.0682}	[0.172, 0.348] {0.0877, 0.176}	[0.203, 0.412] {0.122, 0.245}	[0.217, 0.441]
3	[0.217, 0.440] {0.122, 0.246}	[0.416, 0.898] {0.242, 0.504}	[0.467, 1.04] {0.309, 0.664}	[0.490, 1.11]
4	[0.403, 0.847] {0.242, 0.504}	[0.678, -1] {0.422, 0.992}	[0.743, -1] {0.517, 1.75}	[0.771, -1]

$[r_0, \bar{r}_0]$  for  $V_{k,n}^{\mathbb{C}}$  and  $\{r_0, \bar{r}_0\}$  for  $G_{k,n}^{\mathbb{C}}$  with respect to (35) for  $R = 10$

(-1 in the tables means, that Maple could not find a solution, due to approximation error/too large sphere radii).

Observe that in the lower rate regime  $r_0$  still grows with  $n$  in  $G_{k,n}^{\mathbb{C}}$ , but slowly decreases in  $V_{k,n}^{\mathbb{C}}$ , while in the high rate regime  $r_0$  is strict monotone with respect to  $n$ , expecting the intervals to become disjoint.

So, while the high rate requirement is too restrictive, the results for low rates are unsatisfactory in part. But the general analysis of the lower bound in section IV will come up with interesting results, supporting this approach. To clarify the presentation let us summarize the results so far in the

### Proposition III.1

The inequalities (35) provide approximate bounds on the (geodesic) minimal distance for packings/space time codes on the Stiefel (coherent case) and Grassmann (non-coherent case) manifolds for any admissible  $(k, n)$ . In particular the lower bound in (35) is computational simple and guarantees the existence of corresponding packings/codes.

Especially for the Stiefel manifold these explicitly calculated bounds appear to be new in the context of coherent space time coding.

### B. Comparisons with related results in the literature

Han and Rosenthal [2] obtained bounds on the scaled chordal distance  $\Delta := \frac{d^V}{2\sqrt{k}}$  in the unitary case  $k = n$ ,  $V_{k,k}^{\mathbb{C}} = U(k)$ . Based on a numerically calculated exact volume they extracted three upper bounds. The following table shows their (best) upper bounds (2nd row) for  $\Delta = d^V/(2\sqrt{2})$  in  $U(2)$  (in part relying on the results in [13]) for different rates  $R$  (1st row) together with the upper bounds obtained here (3rd row)

$R$	2.29	2.79	3.0	3.16	3.32	3.45	3.50	4.98
$\Delta$	0.675	0.619	0.597	0.580	0.558	0.542	0.535	0.327
$\frac{r^V}{2}$	1.40	1.01	0.909	0.843	0.785	0.742	0.727	0.409

Note, that equality in the (rough) estimate  $\frac{d^V}{2\sqrt{2}} \leq \frac{r^V}{2}$  (Lemma B.1) has been forced in the third row of the table to convert the geodesic distances computed by a Maple program into chordal distances. Consequently the bounds of [2] are tighter than the bounds obtained here, since in the case of unitary matrices there are more specialized (but less general) methods available to obtain bounds.

As already stated in the introduction another (asymptotic) result has been obtained by Barg and Nogin. For the non-asymptotic case they presented an exact volume formula [1, eq. (11)] for regions in the (real and complex) Grassmann manifold.

$$\begin{aligned} \text{vol } B(r) &= 2^k |G_{k,n}^{\mathbb{C}}| \prod_{i=1}^k \frac{(n-i)!}{[(i-1)!]^2 (n-k-i)!} \times \\ &\quad \int_{0 < \vartheta_1 < \dots < \vartheta_k < \pi/2} \prod_{\|\vartheta\|_2 \leq r} d\vartheta_1 \dots d\vartheta_k \times \\ &\quad \prod_{i=1}^k (\sin \vartheta_i)^{2(n-2k)+1} \cos \vartheta_i \prod_{j < l} (\sin^2 \vartheta_j - \sin^2 \vartheta_l)^2 \end{aligned} \quad (36)$$

which can be computed in polar coordinates, compare A-II. Although (36) is exact, it does not provide a closed form for varying dimensions. Moreover the computations are elaborate compared with the ones done here, such that the evaluation of (29) become intractable.

#### IV. ANALYSIS OF THE LOWER BOUND

The lower bound for the (geodesic) minimal distance  $r_0$  guarantees the existence of corresponding packings/codes. Due to (33) we can explicitly solve the lower bound in (35):

$$\underline{r}_0^{D_{n,k}} = \frac{1}{2^{nR}} \cdot \frac{\text{vol } M}{|B^{D_{n,k}}|} \quad (37)$$

with  $D_{n,k}$  defined as  $D$  in (27), thus

$$D_{n,k} = 2nk - \epsilon k^2, \quad \epsilon = \begin{cases} 1, & V_{k,n}^C \\ 2, & G_{k,n}^C \end{cases} \quad (27')$$

Then

##### Theorem IV.1

The (geodesic) minimal distance  $r_0$  in  $M$  can be lower bounded by

$$\underline{r}_0 \geq \left(\frac{1}{2}\right)^{\frac{nR}{D_{n,k}}} \quad (38)$$

with the right hand side monotonically increasing as a function of  $n$  for  $n \geq k$ . Asymptotically

$$\lim_{n \rightarrow \infty} \underline{r}_0 = \sqrt{\frac{k}{2^{R/k}}} \quad (39)$$

holds.

In particular this establishes a monotonically increasing lower estimate for  $\underline{r}_0$  common for  $V_{k,n}^C$  and  $G_{k,n}^C$ , which is not obvious from the picture drawn from the explicit calculations of  $\underline{r}_0$  for rate  $R = 1$  in the previous sections. Of course, the theorem also holds for the (topological) minimal distance  $\underline{d}_0 = \frac{1}{\alpha} \underline{r}_0$ , connecting this result with space time coding theory.

*Proof:*

Set  $a := 2^{-nR/D_{n,k}}$  and  $b := \left(\frac{\text{vol } M}{|B^{D_{n,k}}|}\right)^{1/D_{n,k}}$ . Then  $\underline{r}_0 = a \cdot b$  and from (27')  $a$  is monotonically increasing as a function of  $n$  with  $\lim_{n \rightarrow \infty} a = 2^{-R/2k}$ .

For  $b$  we show

$$b \geq 1 \quad (40)$$

and

$$\lim_{n \rightarrow \infty} b = \sqrt{k} \quad (41)$$

and the theorem follows.

For the two cases of interest  $b^{D_{n,k}}$  is given as (using (A.10),(11),(19))

$$b^{D_{n,k}} = \begin{cases} \frac{\text{vol } V_{k,n}^C}{|B^{k(2n-k)}|} = (2\sqrt{\pi})^k \frac{\Gamma(k(2n-k)/2 + 1)}{\prod_{i=n-k+1}^n \Gamma(i)} \\ \frac{\text{vol } G_{k,n}^C}{|B^{2k(n-k)}|} = \frac{\Gamma(2k(n-k)/2 + 1) \prod_{j=1}^k \Gamma(j)}{\prod_{i=n-k+1}^n \Gamma(i)} \end{cases} \quad (42)$$

for  $n \geq k$ , resp.  $n \geq k + 1$ . The proof of (40) relies on the simple estimate

$$\begin{aligned} \frac{\Gamma(M+1)}{\Gamma(m+1)} &= (m+1)(m+2) \cdot \dots \cdot M \\ &\geq (m+1)^{M-m} \geq m^{M-m} \end{aligned} \quad (43)$$

for  $m, M \in \frac{1}{2}\mathbb{N}, M - m \in \mathbb{N}$ . Since  $D_{n,k} > 0$  it suffices to show  $B_{k,n} := b^{D_{n,k}} \geq 1$  for all admissible  $(k, n)$ . This will be proven by induction over  $k$  and  $n$ .

$V_{k,n}^C$ :

- 1)  $B_{1,1} = 2\sqrt{\pi} \Gamma(3/2) = \pi > 1$
- 2) Induction over  $k$

$$\begin{aligned} \frac{B_{k+1,k+1}}{B_{k,k}} &= 2\sqrt{\pi} \frac{\overbrace{\Gamma(k^2/2 + (k+1)/2 + 1)}^M}{\Gamma(k^2/2 + 1) \Gamma(k+1)} \\ &\stackrel{(43)}{\geq} 2\sqrt{\pi} \frac{(k^2/2 + 1)^{k+1/2}}{k!} \\ &\stackrel{k^2/2+1>k}{\geq} 2\sqrt{\pi} \sqrt{k^2/2 + 1} \\ &\stackrel{k=1}{\geq} \sqrt{6\pi} > 1 \end{aligned}$$

Thus  $B_{k,k} > 1 \forall_{k \geq 1}$

- 3) Induction over  $n \geq k$

$$\begin{aligned} \frac{B_{k,n+1}}{B_{k,n}} &= \frac{\Gamma(k/2(2n-k) + k + 1)}{\underbrace{\Gamma(k/2(2n-k) + 1)}_m} \frac{\overbrace{\Gamma(n-k+1)}^{\tilde{m}}}{\Gamma(n+1)} \\ &\stackrel{(43)}{\geq} \frac{(m+1) \cdot \dots \cdot (m+k)}{(\tilde{m}+1) \cdot \dots \cdot (\tilde{m}+k)} \geq 1 \end{aligned}$$

since  $m - \tilde{m} = n(k-1) - k^2/2 + k \stackrel{n \geq k}{\geq} k^2/2 > 0$ .

Thus for every  $k \geq 1$  we have  $B_{k,n} > 1 \forall_{n \geq k}$ .

$G_{k,n}^C$ :

- 1)  $B_{1,2} = \frac{\Gamma(2)\Gamma(1)}{\Gamma(2)} = 1$
- 2) Induction over  $k$

$$\frac{B_{k+1,k+2}}{B_{k,k+1}} = \frac{\Gamma(k+2)}{\Gamma(k+1)} \Gamma(k+1) \frac{\prod_{i=2}^{k+1} \Gamma(i)}{\prod_{i=2}^{k+2} \Gamma(i)} = 1$$

Thus  $B_{k,k+1} = 1 \forall_{k \geq 1}$

- 3) Induction over  $n \geq k + 1$

$$\begin{aligned} \frac{B_{k,n+1}}{B_{k,n}} &= \frac{\Gamma(k(n-k) + k + 1)}{\underbrace{\Gamma(k(n-k) + 1)}_m} \frac{\overbrace{\Gamma(n-k+1)}^{\tilde{m}}}{\Gamma(n+1)} \\ &\stackrel{(43)}{=} \frac{(m+1) \cdot \dots \cdot (m+k)}{(\tilde{m}+1) \cdot \dots \cdot (\tilde{m}+k)} \geq 1 \end{aligned}$$

since  $m - \tilde{m} = n(k-1) - k^2/2 + k \stackrel{n \geq k+1}{\geq} k-1 \geq 0$

and it follows for every  $k \geq 1$ , that  $B_{k,n} \geq 1 \forall_{n \geq k+1}$  as desired.

Let us now prove (41). At first  $(2\sqrt{\pi})^{k/D_{n,k}} \xrightarrow{n \rightarrow \infty} 1$  and  $\prod_{j=1}^k \Gamma(j)^{1/D_{n,k}} \xrightarrow{n \rightarrow \infty} 1$  holds. So it remains the evaluation

of

$$\lim_{n \rightarrow \infty} \left( \frac{\Gamma(D_{n,k}/2 + 1)}{\prod_{i=n-k+1}^n \Gamma(i)} \right)^{1/D_{n,k}}$$

Stirling's formula reads either ('~' denotes asymptotic equivalence)

$$\Gamma(m+1) \sim \sqrt{2\pi m} \left( \frac{m}{e} \right)^m \quad \text{or} \quad \Gamma(m) \sim \sqrt{\frac{2\pi}{m}} \left( \frac{m}{e} \right)^m$$

and by  $D_{n,k} \sim 2nk$  we deduce

$$\begin{aligned} \left( \frac{\Gamma(D_{n,k}/2 + 1)}{\prod_{i=n-k+1}^n \Gamma(i)} \right)^{1/D_{n,k}} &\sim \left( \frac{\sqrt{2\pi D_{n,k}/2} \sqrt{\frac{D_{n,k}}{2e}}}{\prod_{i=n-k+1}^n \sqrt{\frac{2\pi}{i}} \left( \frac{i}{e} \right)^i} \right)^{1/D_{n,k}} \\ &= \underbrace{\sqrt{\frac{\pi}{(2\pi)^k}}}_{\rightarrow 1} \underbrace{\frac{1}{\sqrt{2e}}}_{\rightarrow 1} \underbrace{\frac{\sqrt{D_{n,k}}}{\left[ \prod_{i=n-k+1}^n \frac{1}{\sqrt{i}} \left( \frac{i}{e} \right)^i \right]^{1/D_{n,k}}}}_{\rightarrow 1} \\ &\sim \underbrace{\frac{1}{\sqrt{2e}}}_{\rightarrow 1} \underbrace{\frac{1}{\left( \prod_{i=n-k+1}^n \frac{1}{\sqrt{i}} \right)^{1/D_{n,k}}}}_{\rightarrow 1} \times \\ &\quad \underbrace{\sqrt{D_{n,k}} e^{-1/D_{n,k} \sum_{i=n-k+1}^n i \ln \frac{i}{e}}}_{\rightarrow \sqrt{2ek}} \\ &\sim \sqrt{k} \end{aligned}$$

This proves (41)  $\blacksquare$

#### A. Final remarks and application to coding theory

A remarkable coincidence arises from Barg/Nogin's results for the chordal distance in  $G_{k,n}^C$ . Denoting the lower bound in (5) by  $\underline{d}_0$  we find

$$\underline{d}_0 = \lim_{n \rightarrow \infty} \underline{r}_0 \quad (44)$$

therefore, the geodesic lower bound  $\underline{r}_0$  obtained from the flat geodesic volume estimate  $v^0(r) \leq v(r)$  asymptotically equals the (asymptotic) exact chordal lower bound (5). This seems reasonable since in flat space the geodesic distance coincides with the Euclidean (chordal) distance.

Apart from the asymptotics, let us consider the lower bound (38) of Theorem IV.1. It guarantees the existence of packings with minimal distance  $r_0$  bounded monotonically from below in  $V_{k,n}^C$ , resp.  $G_{k,n}^C$ , when  $n$  grows. In coding theory  $(\widetilde{V}_{k,n}^C = \sqrt{\frac{n}{k}} V_{k,n}^C, d^V)$ , resp.  $(\widetilde{G}_{k,n}^C = \sqrt{\frac{n}{k}} G_{k,n}^C, d^G)$ , represent the coding spaces for space time block codes for the Rayleigh flat fading channel unknown to the transmitter and known, resp. unknown, channel at the receiver. The factor  $\sqrt{\frac{n}{k}}$  serves as a constraint, holding the transmit power at each time step constant for different choices of  $(k, n)$ , thus provide a fair comparison of codes from different coding spaces. In a Riemannian manifold  $M$  with metric  $g$  the mapping  $(\lambda M, g) \mapsto (M, \lambda^2 g)$ ,  $\lambda > 0$ , is isometric, leading immediately to the scaled geodesic minimal distance  $\tilde{r}_0 = \lambda r_0$ .

With respect to the coding distances  $d$  we obtain instead

$$\begin{aligned} \left( \sqrt{\frac{n}{k}} M, d \right) &\cong \left( M, \sqrt{\mu \frac{n}{k}} d \right), \\ \mu := \begin{cases} 1, & M = V_{k,n}^C \\ \frac{1}{2}, & M = G_{k,n}^C \end{cases} \end{aligned} \quad (45)$$

whereas  $\frac{\rho}{4} \mu \frac{n}{k}$  ( $\rho \geq 1$  denoting the signal to noise ratio) is (a lower bound of) the first order term (the so called diversity sum, our metric here) in the expansion of the Chernov bound for the pairwise error probability, compare [4, formulas (17)(18)(19)(20)]: The factor of  $\frac{1}{2}$  for  $G_{k,n}^C$  stems from the slightly different 'effective' transmit power  $\underline{\varrho} := \frac{(\rho n/k)^2}{4(1+\rho n/k)}$  compared to the known channel effective transmit power  $\bar{\varrho} := \frac{\rho n}{4k}$ , satisfying  $\frac{1}{2}\bar{\varrho} \leq \underline{\varrho} \leq \bar{\varrho}$ , whereas  $\rho \geq 1, n \geq k$  is understood. Collecting all formulas we finally infer from Theorem IV.1:

#### Corollary IV.2

Given  $\rho \geq 1$  and  $n \geq 2k$ , there exist space time block codes with minimal distance  $\tilde{d}_0$  lower bounded by

$$\tilde{d}_0 = \frac{\sqrt{\mu}}{\alpha} \sqrt{\frac{n}{k}} \underline{r}_0 \geq \frac{\sqrt{\mu}}{\alpha} \sqrt{\frac{n}{k}} \left( \frac{1}{2} \right)^{\frac{nR}{D_{n,k}}} \quad (46)$$

whereas  $\alpha$  is determined by (15), resp. (25),  $D_{n,k}$  is defined in (27) (resp. (27')), and  $\mu$  in (45).

Thus the performance (which scales with  $\tilde{d}_0^2$ ) potentially increases monotonically at least proportionally to  $\frac{n}{k}$ .

The last statement in the corollary follows from the observation, that the diversity (essentially the inverse of the Chernov bound for the pairwise error probability) as a basic performance measure for space time codes [4] is a homogenous polynomial. The first order term coincides with the metric  $\tilde{d}^2$ , while all higher order terms scale with a power of  $\tilde{d}^2$  when code design is interpreted as a constrained packing problem (considering the higher order terms as constraints according to a normalized distance distribution).

## V. CONCLUSIONS

The framework in [1], [2] had been successfully generalized to the Stiefel manifold  $V_{k,n}^C$ ,  $n \geq k$ , and to  $G_{k,n}^C$ ,  $\infty \gg n \geq k/2$  using the completely different method of Riemannian volume bounds (Proposition III.1). Unlike the exact volume formula the lower bound can be relatively simple analyzed as a function of  $(k, n)$  for both  $G_{k,n}^C$  and  $V_{k,n}^C$ , leading to Theorem IV.1, resp. Corollary IV.2. Although the used estimates were quite conservative they apply (in principle) in any Riemannian homogeneous spaces.

The connection to the coding theory of space time block codes advocates further efforts in finding codes in the spaces  $V_{k,n}^C$ , resp.  $G_{k,n}^C$  for  $n$  much larger than  $k$ . Since the minimal distances grow proportionally to  $\sqrt{\frac{n}{k}}$  while the transmit power per time step remains constant, there is a considerable performance impact to expect, when coding in  $V_{k,n}^C$  (resp.  $G_{k,n}^C$ ) as

opposed to coding in  $U(k)$ <sup>3)</sup>. Furthermore, as already pointed out in the introduction the developments in space frequency coding indicate, that the relevant coding spaces are subsets in some  $V_{k,n}^{\mathbb{C}}$  (resp.  $G_{k,n}^{\mathbb{C}}$ ) whereas the number of subcarriers  $n$  satisfies  $n \gg k$ , thus the results proven here may apply to space frequency codes as well.

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### APPENDIX A DIFFERENTIAL GEOMETRIC CALCULATIONS

#### I. $U(n)$ -normal homogeneous spaces

For the theoretical background, common notation and curvature formulas I refer to [15] as a reference.

1) *The unitary group:*

$$U(n) = \{\bar{\Phi} \in \mathbb{C}^{n \times n} \mid \bar{\Phi}^\dagger \bar{\Phi} = 1\} \quad (\text{A.1})$$

is a compact, connected Lie group and a real manifold of dimension  $\dim_{\mathbb{R}} = n^2$ . The corresponding Lie algebra (i.e. the tangent space of  $U(n)$  at 1) is

$$\mathfrak{u}(n) = \{\bar{X} \in \mathbb{C}^{n \times n} \mid \bar{X}^\dagger = -\bar{X}\} \quad (\text{A.2})$$

and the matrix exponential  $\exp$  maps  $\mathfrak{u}(n)$  into  $U(n)$ . On  $\mathfrak{u}(n)$  the (bi-invariant) Riemannian metric for  $U(n)$  is defined as

$$\langle \bar{X}, \bar{Y} \rangle = \frac{1}{2} \operatorname{tr} \bar{X}^\dagger \bar{Y} \quad (\text{A.3})$$

thus  $\langle \bar{X}, \bar{X} \rangle = \frac{1}{2} \|\bar{X}\|_{\text{F}}^2$  (Frobenius norm) holds.

A manifold  $M$  is a  $U(n)$ -homogeneous space, if there is a transitive  $U(n)$  action on  $M$  such that  $M \cong U(n)/H$  for some isotropy subgroup  $H \subset U(n)$ . If  $\mathfrak{h} \subset \mathfrak{u}(n)$  denotes the Lie algebra of  $H$  there is a canonical decomposition of tangent vectors  $\mathfrak{h} \oplus \mathfrak{h}^\perp = \mathfrak{u}(n) \ni \bar{X} = X^\parallel + X$  and we can identify tangents of  $M$  with so called 'horizontal' tangent vectors  $X \in \mathfrak{h}^\perp$ . With this identification  $M$  is called normal homogeneous.

Then the sectional curvature  $K$  and the Ricci curvature  $\operatorname{Ric}$  of  $M$  are given as

$$K(X, Y) = \frac{1}{4} \|[X, Y]\|^2 + \frac{3}{4} \|[X, Y]^\parallel\|^2 \quad (\text{A.4})$$

$$\operatorname{Ric}(e_i, e_i) = \sum_j K(e_i, e_j) \quad (\text{A.5})$$

whereas  $[X, Y] = XY - YX$ ,  $X$  and  $Y$  are normalized tangent vectors and  $\{e_i\}$  denotes a orthonormal base in  $\mathfrak{h}^\perp$ . Note that the sectional curvature  $K$  is always non-negative

$$K \geq 0 \quad (\text{A.6})$$

<sup>3)</sup>Note that this does not contradict the (converse) conclusions in [14], which do not apply here: The error probability computations done there with respect to increasing block length  $n \rightarrow \infty$  are constrained by a fixed total number of code symbols sent. This is a different scenario, not relevant for the analysis performed here.

2) *Supplements for the Stiefel manifold:* The (complex) Stiefel manifold (8) is canonically a  $U(n)$ -normal homogeneous space: The canonical left multiplication of  $k$ -frames in  $\mathbb{C}^n$  by unitary  $n \times n$  matrices transforms each pair of  $k$ -frames into each other. Thus the group action of  $U(n)$  on  $V_{k,n}^{\mathbb{C}}$  is transitive with isotropy group  $H = \begin{pmatrix} \mathbb{1} & \mathbf{0} \\ \mathbf{0} & U(n-k) \end{pmatrix}$  and establishes the canonical diffeomorphism (9). Then  $\mathfrak{h} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathfrak{u}(n-k) \end{pmatrix}$  and tangents  $X \in \mathfrak{h}^\perp$  have the form (12), and (13) follows for the geodesic distance  $r^V$ . Note that this distance is not induced by the length of the geodesics obtained from the canonical embedding of  $V_{k,n}^{\mathbb{C}}$  into  $\mathbb{C}^{n \times k}$ , compare [9] in the real case and additionally [15, Example 6.61(b)] in the complex case.

3) *Supplements for the Grassmann manifold:* The (complex) Grassmann manifold (16) carries the structure of a  $U(n)$ -normal homogeneous space by forgetting not only the orthogonal complement of  $\bar{\Phi} \in U(n)$  (which has been done for  $V_{k,n}^{\mathbb{C}}$ ) but also the particular choice of the spanning  $k$ -frame. Thus  $H = \begin{pmatrix} U(k) & \mathbf{0} \\ \mathbf{0} & U(n-k) \end{pmatrix}$  and this leads to (17). Note that the coordinate representation  $\langle \Phi \rangle \cong \Phi \Phi^{1-1}$  holds only locally in general, but it turns out, that this representation covers all but a set of measure zero, hence we abandon this distinction between local and global properties in this work and drop the distinction between  $G_{k,n}^{\mathbb{C}}$  and its coordinate domain. Calculating  $\mathfrak{h}^\perp$  leads to tangents of the form (20).

Given two elements  $\langle \Phi \rangle, \langle \Psi \rangle \in G_{k,n}^{\mathbb{C}}$  then the  $k$  stationary angles  $0 \leq \vartheta_1 \leq \dots \leq \vartheta_k \leq \pi/2$  between  $\langle \Phi \rangle$  and  $\langle \Psi \rangle$  are defined successively by the critical values  $\arccos|\langle v_i, w_i \rangle|$ ,  $i = 1, \dots, k$  (in increasing order), of  $(v, w) \mapsto \arccos|\langle v, w \rangle|$  where the unit vectors  $v, w$  vary over  $\{v_1, \dots, v_{i-1}\}^\perp \subset \langle \Phi \rangle$ , respectively  $\{w_1, \dots, w_{i-1}\}^\perp \subset \langle \Psi \rangle$ . It is well known that the stationary angles can be computed by the formula (any representing  $k$ -frame will do)

$$\cos \vartheta_i = \sigma_i(\Phi^\dagger \Psi), \quad i = 1, \dots, k \quad (\text{A.7})$$

whereas  $\sigma_i(M)$ ,  $i = 1, \dots, k$  denotes the  $i$ -th singular value of the matrix  $M$  in decreasing order.

Given a tangent  $X = \begin{pmatrix} \mathbf{0} & -B^\dagger \\ B & \mathbf{0} \end{pmatrix}$ ,  $B \in \mathbb{C}^{(n-k) \times k}$ , the singular value decomposition  $B = V \Sigma W^\dagger = V_1 S W^\dagger$ ,  $V = (V_1, V_2) \in U(n-k)$ ,  $W \in U(k)$ ,  $\Sigma = \begin{pmatrix} S & \mathbf{0} \end{pmatrix}$ ,  $S = \operatorname{diag}(\sigma_1, \dots, \sigma_k)$ , yields  $X = U \Delta U^\dagger$  with  $U = \begin{pmatrix} W & \mathbf{0} \\ \mathbf{0} & V_1 V_2 \end{pmatrix}$ ,  $\Delta = \begin{pmatrix} \mathcal{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ , and  $\mathcal{D} = \begin{pmatrix} \mathbf{0} & -S \\ S & \mathbf{0} \end{pmatrix}$ . From this one calculates  $(\exp X)(\begin{pmatrix} \mathbf{1} & \mathbf{0} \end{pmatrix}) = \begin{pmatrix} W(\cos S)W^\dagger \\ V_1(\sin S)W^\dagger \end{pmatrix}$ , and  $\cos \vartheta = \sigma((\mathbf{1} \mathbf{0})(\exp X)(\begin{pmatrix} \mathbf{1} & \mathbf{0} \end{pmatrix})) = \cos S$ , thus  $\vartheta = \sigma$  and (21) follows.

The space of orthogonal projections  $\Pi_V := \{P_\Phi \mid \Phi \in V_{k,n}^{\mathbb{C}}\}$  (compare (22)) can be identified with  $G_{k,n}^{\mathbb{C}}$ . In particular we have  $\Pi_V = \Pi_k$  with

$$\begin{aligned} \Pi_k := \{P \in \mathbb{C}^{n \times n} \mid P^\dagger = P, P^2 = P, \operatorname{tr} P = k, \\ \|P - k/n \mathbf{1}\|_{\text{F}}^2 = k(n-k)/n\} \end{aligned} \quad (\text{A.8})$$

as one can see by picking an appropriate representative  $\Phi \in \langle \Phi \rangle$  (e.g.  $\Phi = \begin{pmatrix} \mathbb{1} & \mathbf{0} \end{pmatrix}$  due to invariance of  $\Pi_k$  under the left and right unitary action). Since each  $P \in \Pi_k$  is Hermitian with constant trace,  $\Pi_k$  is canonically a real submanifold of  $\mathbb{R}^{n^2-1}$ , the constant norm justifies the embedding (23)

## II. Volume computations

1) *Total volume*: The unitary group  $U(n) \subset \mathrm{GL}(n, \mathbb{C})$  can be equipped with the induced Lebesgue measure from the ambient space  $\mathbb{R}^{2n^2}$ . The Stiefel manifold inherits its volume measure from its total space  $U(n)$ : We get from the familiar volume formulas

$$|S^{m-1}| := \mathrm{vol} S^{m-1} = \frac{2\pi^{m/2}}{\Gamma(m/2)} \quad (\text{A.9})$$

$$|B^m| := \mathrm{vol} B^m = |S^{m-1}|/m \quad (\text{A.10})$$

for the unit sphere  $S^{m-1}$  and the unit ball  $B^m$  in  $\mathbb{R}^m$  and the canonical homogeneous family  $S^{2m-1} \cong U(m)/U(m-1)$  the following recursive formula  $\mathrm{vol} U(1) = |S^1| = 2\pi$ ,  $\mathrm{vol} U(m) = |S^{2m-1}| \mathrm{vol} U(m-1)$ , and therefore (11) and (19).

2) *Volume for regions in  $G_{k,n}^{\mathbb{C}}$* : The volume formula for regions in the complex Grassmann manifold will be derived, based on [1, eq. (11)]<sup>4)</sup>.

Starting with formula [16, (A.18)] for the distribution of eigenvalues  $\lambda_i$ , with  $\lambda_i = \cos^2 \vartheta_i$  of  $(\mathbf{1}, \mathbf{0}) \Phi \Psi^\dagger (\mathbf{0})$  we obtain the volume density as the marginal density

$$\begin{aligned} \omega &= C(k, n) \cdot \prod_{i=1}^k (1 - \lambda_i)^{n-2k} \prod_{j < l} (\lambda_l - \lambda_j)^2 \cdot d\lambda_1 \dots d\lambda_k \\ &= C(k, n) \cdot 2^k k! \cdot \prod_{i=1}^k (\sin \vartheta_i)^{2(n-2k)+1} \cos \vartheta_i \\ &\quad \prod_{j < l} (\sin^2 \vartheta_j - \sin^2 \vartheta_l)^2 \cdot d\vartheta_1 \dots d\vartheta_k \end{aligned} \quad (\text{A.11})$$

whereas the Jacobi determinant  $2^k \prod_{i=1}^k \sin \vartheta_i \cos \vartheta_i$  of the mapping  $\lambda \mapsto \vartheta$  has been introduced in order to express the volume density in terms of  $\vartheta$ , and  $k!$  establishes the ordering condition on the (open) simplex  $\Theta = \{0 < \vartheta_1 < \dots < \vartheta_k < \pi/2\}$  of stationary angles. The constant  $C$  is just a normalization factor, which reads in our case

$$C(k, n) = \frac{|G_{k,n}^{\mathbb{C}}|}{k!} \prod_{i=1}^k \frac{(n-i)!}{[(i-1)!]^2 (n-k-i)!} \quad (\text{A.12})$$

(without the factor  $|G_{k,n}^{\mathbb{C}}|$  this would give the Haar measure used in [1] on  $G_{k,n}^{\mathbb{C}}$ ). The volume of sufficiently small geodesic balls is now given as

$$\begin{aligned} \mathrm{vol} B(r) &= \int_{\Theta \cap \{\|\vartheta\|_2 \leq r\}} \omega(\vartheta) \\ &= \int_0^r d\rho \int_{\alpha, \beta \in [0, \pi/2]} \frac{1}{k!} \omega(\vartheta(\rho, \alpha, \beta)) |\det J_\vartheta(\rho, \alpha, \beta)| \\ &\quad d\alpha d\beta_1 \dots d\beta_{k-2} \end{aligned} \quad (\text{A.13})$$

<sup>4)</sup>Unfortunately (in their first paper version) their formula is not correct in the complex case (private communication). Fortunately this does not affect the (asymptotic) results obtained in [1]. An erratum has already been produced, thus the derivation here is only for completeness of the presentation and the convenience of the reader

whereas  $(\rho, \alpha, \beta)$  denote ( $k$  dimensional) polar coordinates

$$\begin{aligned} \vartheta_1 &= \rho \cos \beta_{k-2} \dots \cos \beta_1 \cos \alpha \\ \vartheta_2 &= \rho \cos \beta_{k-2} \dots \cos \beta_1 \sin \alpha \\ &\quad \dots \\ \vartheta_{k-1} &= \cos \beta_{k-2} \sin \beta_{k-3} \\ \vartheta_k &= \sin \beta_{k-2} \end{aligned} \quad (\text{A.14})$$

The factor  $\frac{1}{k!}$  removes the ordering condition on the simplex, such that the domain of angle integration is the whole region  $[0, \pi/2]^{k-1}$ . Eventually,  $J_\vartheta$  denotes the Jacobi matrix of the coordinate transformation  $(\rho, \alpha, \beta) \mapsto \vartheta$ .

## APPENDIX B THE LOCAL EQUIVALENCE OF $d$ AND $r$ IN $V_{k,n}^{\mathbb{C}}$

In this appendix the proof of Proposition II.1 will be carried out. Let us recall, what we want to show. Given  $\Phi, \Psi$  in the complex Stiefel manifold  $V_{k,n}^{\mathbb{C}} \subset \mathbb{C}^{n \times k}$ , the topological distance  $d$  motivated from coding theory is given as  $d = \|\Phi - \Psi\|_{\mathrm{F}}$  (we drop the upper index ' $V$ ' in this appendix).

At the same time, locally there is a unique geodesic  $\gamma$  in  $V_{k,n}^{\mathbb{C}}$  joining  $\Phi$  and  $\Psi$ , and the geodesic distance  $r$  is simply defined as its length  $L = \int_0^1 \|\dot{\gamma}(t)\|_{\mathrm{F}} dt$ ,  $\dot{\gamma}(t)$  being the parallel transported horizontal tangent vector  $X(\gamma(t))$  along  $\gamma$ . Thus we obtain  $r = \|X(\gamma(0))\|_{\mathrm{F}}$ . Since both  $d$  and  $r$  are invariant under the action of the  $U(n)$  we can set  $\Psi = (\mathbf{1}, \mathbf{0})$  without loss of generality. Recalling the general form  $X = \begin{pmatrix} A & -B^\dagger \\ B & \mathbf{0} \end{pmatrix}$ ,  $A \in \mathfrak{u}(k)$ ,  $B \in \mathbb{C}^{(n-k) \times k}$ , of horizontal tangent vectors in  $u(n)$  (12) we arrive at

$$d^2 = \|\Phi - (\mathbf{1}, \mathbf{0})\|_{\mathrm{F}}^2 \quad (\text{B.1})$$

$$r^2 = \frac{1}{2} \|X\|_{\mathrm{F}}^2 = \frac{1}{2} \|A\|_{\mathrm{F}}^2 + \|B\|_{\mathrm{F}}^2 \quad (\text{B.2})$$

whereas  $\Phi = \exp X (\mathbf{1}, \mathbf{0})$ . Unlike the case  $A = 0$  (representing tangents for  $G_{k,n}^{\mathbb{C}}$ ) there is no closed form expression for  $\Phi$  in terms of  $X$  in general (compare [9]), so it remains a non-trivial task to find constants  $\alpha, \beta > 0$  satisfying

$$\boxed{\beta d \leq r \leq \alpha d} \quad (\text{15}')$$

expressing the equivalence of  $d$  and  $r$ .

To begin with the easy cases, the constant  $\beta$  is easily found, as well as  $\alpha$  when  $k = n$ : Both are simple consequences of the two sided inequality  $\sin x \leq x \leq (\pi/2) \sin x$ , whereas  $x \in [0, \pi/2]$  is understood in the second inequality.

### Lemma B.1

In  $V_{k,n}^{\mathbb{C}}$   $\frac{1}{\sqrt{2}}d \leq r$  always holds, thus we have  $\beta = \frac{1}{\sqrt{2}}$ .

*Proof:*

Since  $X \in \mathfrak{u}(n)$  there exist  $V \in U(n)$  such that  $X = V \mathrm{diag}(\iota\xi) V^\dagger$ , thus  $r^2 = \frac{1}{2} \|\xi\|^2$ ,  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ . Now we can estimate as follows

$$\begin{aligned} d^2 &= \|(\mathbf{1} - \exp X) (\mathbf{1}, \mathbf{0})\|_{\mathrm{F}}^2 \leq \|\mathbf{1} - \exp X\|_{\mathrm{F}}^2 \\ &= \|\mathbf{1} - \exp(\mathrm{diag}(\iota\xi))\|_{\mathrm{F}}^2 = \sum_j |1 - e^{\iota\xi_j}|^2 \\ &= 2 \sum_j (1 - \cos \xi_j) = 4 \sum_j \sin^2 \frac{\xi_j}{2} \leq \|\xi\|^2 = 2r^2 \end{aligned}$$

(since  $\sin^2 x/2 \leq x^2/4$ )

**Lemma B.2**

If  $k = n$  then  $r \leq \frac{\pi}{2\sqrt{2}}d$  holds, thus  $\alpha = \frac{\pi}{2\sqrt{2}}$ .

*Proof:*

$k = n$  implies  $B = 0$ ,  $X = A$  and we can estimate

$$d^2 = \|\mathbf{1} - \exp A\|_{\text{F}}^2 = 4 \sum_j \sin^2 \frac{a_j}{2} \geq \frac{4}{\pi^2} \|a\|^2 = \frac{8}{\pi^2} r^2$$

(since  $x^2/4 \leq (\pi^2/4) \sin^2 x/2$  for  $x \in [-\pi, \pi]$ ), whereas  $\iota a = (\iota a_1, \dots, a_n)$  denotes the vector of eigenvalues of  $A \in \mathfrak{u}(n)$ .  $\blacksquare$

The non-trivial task is to obtain some  $\alpha > 0$ , when  $k < n$ . The rest of this section deals with this job. Let us assume  $k \leq \frac{n}{2}$  since this is the relevant case for the analysis in this work (the case  $k > \frac{n}{2}$  should be similar).

Let  $X = Y + Z$  with  $Y = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ , and  $Z = \begin{pmatrix} \mathbf{0} & -B^\dagger \\ B & \mathbf{0} \end{pmatrix}$ , then we can write

$$\bar{\Phi} = \exp X = (\exp Z) \begin{pmatrix} v & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} \quad (\text{B.3})$$

since this is merely a factorization of  $\Phi = \bar{\Phi} \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}$  into a certain projection onto  $G_{k,n}^{\mathbb{C}}$  and the remaining 'phase' in  $U(k) \ni v$ . The first factor  $\exp Z$  can be calculated in closed form:  $B$  has a singular value decomposition  $B = V \text{diag}(\vartheta^\downarrow) u$  for some  $V \in U(n-k)$ ,  $u \in U(k)$  and  $\vartheta^\downarrow := (\vartheta_k, \dots, \vartheta_1)$  denotes the vector of principal angles (in decreasing order) between  $\langle \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} \rangle$  and  $\langle \Phi \rangle$ . Setting  $U_{\vartheta^\downarrow} = \begin{pmatrix} \text{diag}(\cos \vartheta^\downarrow) - \text{diag}(\sin \vartheta^\downarrow) & \mathbf{0} \\ \text{diag}(\sin \vartheta^\downarrow) & \text{diag}(\cos \vartheta^\downarrow) \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}$ , we arrive at  $\exp Z = \begin{pmatrix} u & \mathbf{0} \\ \mathbf{0} & V \end{pmatrix} U_{\vartheta^\downarrow} \begin{pmatrix} u & \mathbf{0} \\ \mathbf{0} & V \end{pmatrix}^\dagger$ . So we have achieved a quite explicit representation of  $\bar{\Phi}$ . In particular the principal  $k \times k$ -submatrix  $\phi = (\mathbf{1} \ \mathbf{0}) \bar{\Phi} \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}$  reads

$$\phi = u \text{diag}(\cos \vartheta^\downarrow) u^\dagger v \quad (\text{B.4})$$

Now we can start estimating:

$$\begin{aligned} d^2 &= \|\Phi - \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}\|_{\text{F}}^2 = 2(k - \Re \text{tr } \phi) \\ &= 2 \left( k - \sum_{j=1}^k \Re(u^\dagger v u)_{jj} \cos \vartheta_{k-j} \right) \\ &\geq 2 \left( k - \frac{1}{2} \sum_j [\Re(u^\dagger v u)_{jj}]^2 - \frac{1}{2} \sum_j \cos^2 \vartheta_j \right) \end{aligned} \quad (\text{B.5})$$

Writing  $U(k) \ni v = \exp \tilde{A}$ ,  $\tilde{A} \in \mathfrak{u}(k)$  with eigenvalues  $\iota \tilde{a} = \iota(\tilde{a}_1, \dots, \tilde{a}_k)$  of  $\tilde{A}$  we have  $\Re(u^\dagger v u)_{jj} \geq 0$  whenever  $\tilde{a} \in [-\pi/2, \pi/2]^k$ . Demanding this mild locality restriction we get  $[\Re(u^\dagger v u)_{jj}]^2 \leq \Re(u^\dagger v u)_{jj}$ , thus  $\sum_j [\Re(u^\dagger v u)_{jj}]^2 \leq \Re \text{tr}(u^\dagger v u) = \Re \text{tr } v = \sum_j \cos \tilde{a}_j$  and therefore

$$d^2 \geq 2 \sum_j \sin^2 \frac{\tilde{a}_j}{2} + \sum_j \sin^2 \vartheta_j \geq \frac{2}{\pi^2} \|\tilde{a}\|^2 + \frac{4}{\pi^2} \|\vartheta\|^2 \quad (\text{B.6})$$

All what remains to do in order to compare  $d$  with  $r$  is to find the link between  $\tilde{A}$  and  $A$ , respectively  $\tilde{Y} = \begin{pmatrix} \tilde{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$  and  $Y = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ . By (B.3)

$$\exp \tilde{Y} = \begin{pmatrix} v & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} = \exp(-Z) \exp X \quad (\text{B.7})$$

holds, thus our 'missing link' is given by the Baker-Campbell-Hausdorff formula expressing  $W \in \mathfrak{u}(\mathfrak{n})$  given by  $\exp W = \exp U \exp V$ ,  $(U, V) \in \mathfrak{u}(\mathfrak{n})_e \times \mathfrak{u}(\mathfrak{n})_e$  by

$$\begin{aligned} W &= V + \int_0^1 f(e^{t \text{ad}_U} e^{\text{ad}_V}) U dt \\ &= U + V + \sum_{r=1}^{\infty} \frac{(-1)^r}{r+1} \times \\ &\quad \sum_{\substack{p_1, \dots, p_r \geq 0 \\ q_1, \dots, q_r \geq 0 \\ \forall i=1..r \ p_i+q_i > 0}} \frac{\left( \frac{\text{ad}_U^{p_1}}{p_1!} \circ \frac{\text{ad}_V^{q_1}}{q_1!} \circ \dots \circ \frac{\text{ad}_U^{p_r}}{p_r!} \circ \frac{\text{ad}_V^{q_r}}{q_r!} \right) (U)}{p_1 + \dots + p_r + 1} \end{aligned} \quad (\text{B.8})$$

whereas  $f(z) = \frac{\ln z}{z-1}$  and  $\text{ad}_U : V \mapsto [U, V] = UV - VU$  (see [17] for that particular representation of the BCH formula (Dynkin's formula in their terminology)). The second part of (B.8) is nothing but the term-wise integrated Taylor series expansion of the integrand. Following [17] the domain of definition  $\mathfrak{u}(\mathfrak{n})_e$  is the region of  $\mathfrak{u}(\mathfrak{n})$  in which the tangent map of  $\exp$  is regular. It is the complement of  $\{U \in \mathfrak{u}(\mathfrak{n}) \mid \det(\text{ad}_U - 2\pi i \mathbb{Z}' \mathbf{1}) = 0, \mathbb{Z}' = \mathbb{Z} \setminus \{0\}\}$  in  $\mathfrak{u}(\mathfrak{n})$ . In particular,  $\mathfrak{u}(\mathfrak{n})_e$  contains a connected neighborhood

$$D(\delta_0) = \{U \in \mathfrak{u}(\mathfrak{n}) \mid \|U\|_{\text{F}} \leq \delta_0\} \quad (\text{B.9})$$

of  $\mathbf{0}$ . Specializing to  $W = \tilde{Y}$ ,  $U = -Z$ ,  $V = X = Y + Z$  yields in multi-index notation (thus  $|p| = \sum_i p_i$ ,  $p! = \prod_i p_i!$ )

$$\begin{aligned} \tilde{Y} &= Y + \sum_{r=1}^{\infty} \frac{(-1)^r}{r+1} \times \\ &\quad \sum_{\substack{p=(p_1, \dots, p_r) \geq 0 \\ q=(q_1, \dots, q_r) \geq 0 \\ p+q > 0}} \frac{(-1)^{|p|+1} (\text{ad}_Z^{p_1} \circ \text{ad}_X^{q_1} \circ \dots \circ \text{ad}_Z^{p_r} \circ \text{ad}_X^{q_r})(Z)}{(|p|+1)p!q!} \end{aligned} \quad (\text{B.10})$$

Note that every term contributes at least some factor involving  $A$  (since  $\text{ad}_Z(Z) = 0$ ), hence in the norm estimate

$$\begin{aligned} &\|\text{ad}_Z^{p_1} \circ \text{ad}_X^{q_1} \circ \dots \circ \text{ad}_Z^{p_r} \circ \text{ad}_X^{q_r}(Z)\|_{\text{F}} \\ &\leq 2^{|p|+|q|} \|Z\|_{\text{F}}^{|p|+1} \|X\|_{\text{F}}^{|q|} \\ &\leq 2^{|p|+|q|} \|Z\|_{\text{F}}^{|p|+1} \sum_{i=0}^{|q|} \binom{|q|}{i} \|Y\|_{\text{F}}^i \|Z\|_{\text{F}}^{|q|-i} \end{aligned} \quad (\text{B.11})$$

the term corresponding to  $i = 0$  has no counterpart in (B.10) (resp. it is zero in (B.10) already), therefore with  $\|X\|_{\text{F}} \leq \delta$  (thus  $\|Y\|_{\text{F}}, \|Z\|_{\text{F}} \leq \delta$ ) we can factor out one  $\|Y\|_{\text{F}}$  and estimate

$$\begin{aligned} &\sum_{i=1}^{|q|} \binom{|q|}{i} \|Y\|_{\text{F}}^i \|Z\|_{\text{F}}^{|q|-i} \\ &= \left( \sum_{i=1}^{|q|} \binom{|q|}{i} \|Y\|_{\text{F}}^{i-1} \|Z\|_{\text{F}}^{|q|-i} \right) \|Y\|_{\text{F}} \\ &\leq \left( \sum_{i=1}^{|q|} \binom{|q|}{i} \right) \delta^{|q|-1} \|Y\|_{\text{F}} \leq 2^{|q|} \delta^{|q|-1} \|Y\|_{\text{F}} \end{aligned} \quad (\text{B.12})$$

and the  $k \times k$  principal submatrix of (B.10) of our interest satisfies

$$\tilde{A} = A + C \quad (\text{B.13})$$

$$\|C\|_{\text{F}} \leq \kappa \|A\|_{\text{F}} \quad (\text{B.14})$$

$$\kappa = \sum_{r=1}^{\infty} \frac{1}{r+1} \sum_{\substack{p=(p_1, \dots, p_r) \geq 0 \\ q=(q_1, \dots, q_r) \geq 0 \\ p+q > 0}} \frac{2^{|p|+2|q|} \delta^{|p|+|q|}}{(|p|+1)p!q!} \quad (\text{B.15})$$

It is possible to rewrite (B.15) such that we can prove the convergence of the multi-series, that is existence of  $\kappa$ . given some sub-multi-indices  $p_J, q_J$  corresponding to some  $J \subset \{1, \dots, r\}$  let us set  $\lambda_J := \frac{(2\delta)^{|p_J|}}{(|p_J|+1)p_J!}$  and  $\mu_J := \frac{(4\delta)^{|q_J|}}{q_J!}$ , then (B.15) equals  $\sum_r \frac{1}{r+1} \kappa_r$  with ( $J'$  denotes the set  $\{1, \dots, r\} \setminus J$ )

$$\kappa_r = \sum_{s=0}^r \sum_{\substack{J \subset \{1, \dots, r\} \\ |J|=s}} \left\{ \left( \sum_{\substack{p_J \geq 1 \\ p_{J'} \equiv 0}} \lambda_{p_J} \right) \left( \sum_{q \geq 1} \mu_q \right) + \left( \sum_{p \geq 1} \lambda_p \right) \left( \sum_{\substack{q_J \geq 1 \\ q_{J'} \equiv 0}} \mu_{q_J} \right) \right\} \quad (\text{B.16})$$

Now we can perform a rather rough estimate on the sums. We have  $\sum_{q_J \geq 1} \mu_{q_J} = (e^{4\delta} - 1)^{|J|}$  and  $\sum_{p_J \geq 1} \lambda_{p_J} \leq (e^{2\delta} - 1)^{|J|}$ , therefore (note that the sums in the brackets in (B.16) do not depend on the particular choice of  $J \subset \{1, \dots, r\}$  but only on its cardinality  $|J| = s$ )

$$\begin{aligned} \kappa_r &\leq \sum_{s=0}^r \binom{r}{s} \left\{ (e^{2\delta} - 1)^s (e^{4\delta} - 1)^r + (e^{2\delta} - 1)^r (e^{4\delta} - 1)^s \right\} \\ &\leq (e^{4\delta} - 1)^r e^{2\delta r} + (e^{2\delta} - 1)^r e^{4\delta r} \\ &= [(e^{4\delta} - 1)e^{2\delta}]^r + [(e^{2\delta} - 1)e^{4\delta}]^r \end{aligned} \quad (\text{B.17})$$

It is obvious, that we can choose a  $\delta \leq \delta_0$  sufficiently small, such that  $\kappa_r \leq \frac{1}{(r+1)^r}$  for any given  $t > 0$ , ensuring the convergence of (B.15). Setting in particular  $t = 1$  yields  $\kappa \leq \frac{\pi^2}{6} - 1 < 1$  and we obtain from (B.13)

$$\|\tilde{A}\|_{\text{F}} \geq (1 - \kappa) \|A\|_{\text{F}} \quad (\text{B.18})$$

Now we can proceed further with (B.6):

$$\begin{aligned} d^2 &\geq \frac{2(1 - \kappa)^2}{\pi^2} \|a\|^2 + \frac{4}{\pi^2} \|\vartheta\|^2 \\ &\geq \frac{4(1 - \kappa)^2}{\pi^2} \left( \frac{1}{2} \|A\|_{\text{F}}^2 + \|B\|_{\text{F}}^2 \right) = \frac{4(1 - \kappa)^2}{\pi^2} r^2 \end{aligned} \quad (\text{B.19})$$

and we have proven our final lemma:

### Lemma B.3

If  $k \leq \frac{n}{2}$  there exists a  $\delta < \delta_0$ , such that  $(1 - \kappa) > 0$ , whereas  $\delta_0$  and  $\delta$  are determined by (B.9), resp. (B.17) demanding  $\kappa_r \leq \frac{1}{(r+1)^r}$ . Then locally for  $r = \|X\|_{\text{F}} \leq \delta$  the relation  $r \leq \frac{\pi}{2(1-\kappa)} d$  holds, thus  $\alpha = \frac{\pi}{2(1-\kappa)}$ .

This lemma fills the gap in formula (15). Of course,  $(1 - \kappa) \approx 1$  would be optimal in this situation (observe the loss compared

to  $\alpha$  in Lemma B.2), which can be achieved by setting  $\delta \ll 1$ , with  $\kappa$  decreasing the smaller  $\delta$  has been chosen. Unfortunately, the smaller we choose  $\delta$ , the higher the required corresponding rate  $R$  ensuring the validity of Lemma B.3. For example, to obtain a numerical value of  $R_0 \approx 1.4$  (by formula (35) as a lower bound for the corresponding rate, with  $\delta = r_0$ ), which is still achievable for coding purposes in a practical setting, one needs values of  $\delta \approx 1.25$ , which is quite large in order to apply Lemma B.3, thus the estimates done here are far to rough to accomplish that. The importance of the lemma actually lies in the fact, that it proves the *existence* of some  $\alpha > 0$  in (15). However, numerical simulations indicate that the real world behaves much better than the estimates. The histograms in Fig. 1 display  $1 - \kappa$  drawn from 1000 random samples in  $V_{2,n}^{\mathbb{C}}$ ,  $n = 4, 6, 8$  for  $\delta = 1.25$ : Although there seems to be no rigorous and essentially sharper estimate available than the one performed here, the numerical examples indicate, that under still moderate rate constraints  $1 - \kappa \approx 0.9$  holds, thus  $\alpha = \frac{\pi}{\sqrt{2}(1-\kappa)} \approx \frac{\pi}{0.9\sqrt{2}}$  in (15).

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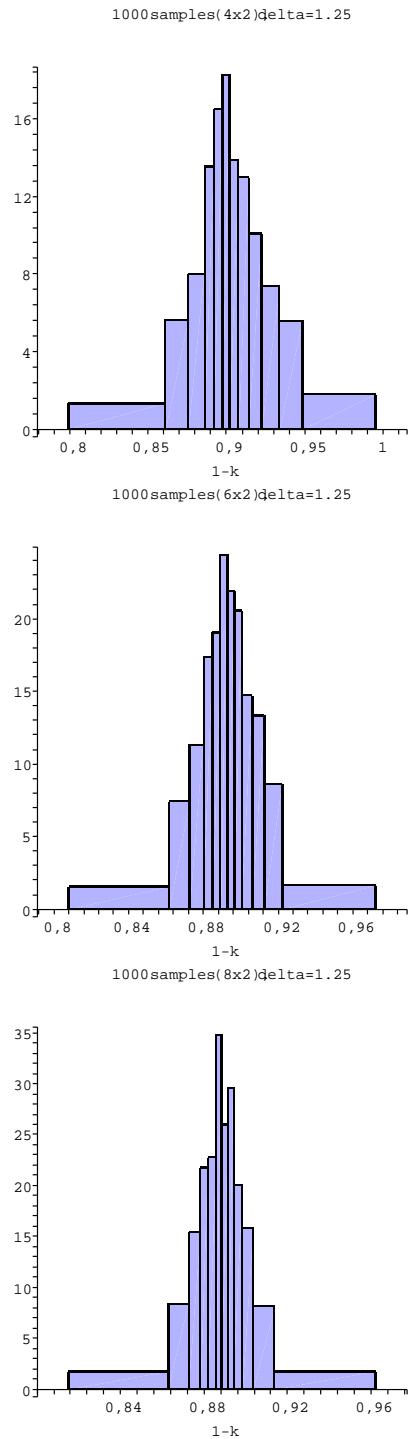


Fig. 1.